United Kingdom
Mathematics Trust

Intermediate Mathematical Olympiad

Maclaurin paper

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. A fruit has a water content by weight of $m\%$. When left to dry in the sun, it loses $(m - 5)\%$ of this water, leaving it with a water content by weight of 50%. What is the value of m?

Solution

Let the original weight be 100. Then $100 - m$ is pulp and m water. After the reduction, there is $100 - m$ pulp and $m\left(1 - \frac{m-5}{100}\right)$ water, and these are equal. Hence $100 - m = \left(1 - \frac{m-5}{100}\right)m$.

Multiplying out, we have $10000 - 100m = 105m - m^2$ and then, rearranging, we obtain the quadratic equation $m^2 - 205m + 10000 = 0$. This factorises as $(m - 125)(m - 80) = 0$. As $m \leq 100$ it follows that $m = 80$.

2. (i) Expand and simplify $(x+1)(x^2-x+1)$.

(ii) Find all powers of 2 that are one more than a cube.

(A power of 2 is a number that can be written in the form 2^n, where n is an integer greater than or equal to 0.)

SOLUTION

(i) $(x+1)(x^2-x+1) = (x^3-x^2+x)+(x^2-x+1) = x^3+1$.

(ii) Suppose that $2^n = k^3+1 = (k+1)(k^2-k+1)$.

Both factors must be positive integers and also powers of 2.

When $n = 0$, we have $2^0 = 1 = 0^3+1$. So 1 is a power of 2 one more than a cube.

When $n = 1$, we have $2^1 = 2 = 1^3+1$. So 2 is another power of 2 one more than a cube.

For all values of $n > 1$, we have $k > 1$, so that both factors $(k+1)$ and (k^2-k+1) are powers of 2 greater than 1. In particular (k^2-k+1) must be even, but this is impossible - k and k^2 have the same parity (either both are even or both are odd), so (k^2-k+1) is always odd.

Hence the only powers of 2 which are one more than a cube are 1 and 2.

3. How many distinct triangles satisfy all the following properties:

(i) all three side-lengths are a whole number of centimetres in length;

(ii) at least one side is of length 10 cm;

(iii) at least one side-length is the (arithmetic) mean of the other two side-lengths?

SOLUTION

By (ii), one side has length 10 cm. Let the lengths of the other two sides be a cm and b cm, which, by (i), are positive integers. Without loss of generality, we assume $a \geq b$. Condition (iii) yields three possibilities.

(a) The mean of a and b is 10, so that $a + b = 20$. Now, in order to respect the triangle inequality, the possibilities for (a, b) are $(10, 10)$, $(11, 9)$, $(12, 8)$, $(13, 7)$ and $(14, 6)$. Note that $(15, 5)$ is impossible since it would result in a degenerate $(15, 10, 5)$ 'triangle'.

(b) The mean of b and 10 is a, so $2a = b + 10$. Now the possibilities are $(10, 10)$, $(9, 8)$, $(8, 6)$ and $(7, 4)$.

(c) The mean of a and 10 is b, so that $2b = a + 10$. Here the possibilities are $(10, 10)$, $(12, 11)$, $(14, 12)$, $(16, 13)$, $(18, 14)$, $(20, 15)$, $(22, 16)$, $(24, 17)$, $(26, 18)$ and $(28, 19)$.

Hence, counting the $(10, 10, 10)$ case only once, there are exactly 17 such triangles.

4. A robot sits at the origin of a two-dimensional plane. Each second the robot chooses a direction, North or East, and at the sth second moves 2^{s-1} units in that direction. The total number of moves made by the robot is a multiple of 3. Show that, for each possible total number of moves, there are at least four different routes the robot can take such that the distance from the origin to the robot's final position is an integer.

SOLUTION

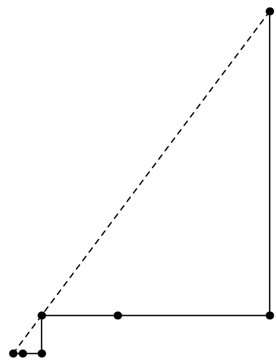
Represent a move East by E and a move North by N.

Suppose the total number of moves is $3k$. We can split this into k triples.

The first triple contains steps of 1, 2 and 4. The routes EEE and NNN result in distances from the origin to the final position of 7. The routes EEN and NNE result in distances of 5, since the moves form a $(3, 4, 5)$ triangle.

For each subsequent triple, the same choices are available.

By repeating the moves in the first triple, we obtain either straight lines or similar triangles, and hence the distances are integers.



An example is shown in the diagram, where the moves are EENEEN. The total displacement East is $1 + 2 + 8 + 16 = 27 = 3 \times 9$ and the total displacement North is $4 + 32 = 36 = 4 \times 9$. The resulting distance from the origin to the final position is $5 \times 9 = 45$. But of course there is no need to do any calculations.

5. The equation $x^2 + bx + c = 0$ has two different integer solutions, and the equation $x^2 + bx - c = 0$ also has two different integer solutions, where b and c are nonzero.

(i) Show that it is possible to find different positive integers p and q such that $2b^2 = p^2 + q^2$.

(ii) Show that it is possible to find different positive integers r and s such that $b^2 = r^2 + s^2$.

SOLUTION

(i) The equations have integer solutions, so by the quadratic formula, there exist non-negative integers p and q so that $b^2 - 4c = p^2$ and $b^2 + 4c = q^2$.

Neither p nor q can equal zero, since then one of the quadratics would have equal roots. As $c \neq 0$, it follows that $p^2 \neq q^2$, so we can take p and q to be *different positive* integers.

Adding, we have $2b^2 = p^2 + q^2$ as required.

(ii) We use the previous result and, without loss of generality, assume $p > q$.

We note that p and q have the same parity — they are either both odd or both even — since otherwise $p^2 + q^2$ would be odd, but $2b^2$ is even.

Now we define $r = \dfrac{p+q}{2}$ and $s = \dfrac{p-q}{2}$, which are *positive* integers, with $p = r + s$ and $q = r - s$.

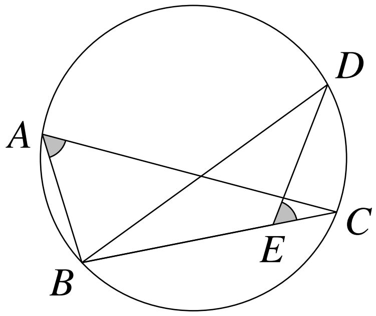
Note finally that $r > s$, so they are different.

It now follows that $p^2 + q^2 = 2r^2 + 2s^2 = 2b^2$, so $b^2 = r^2 + s^2$ as required.

REMARK

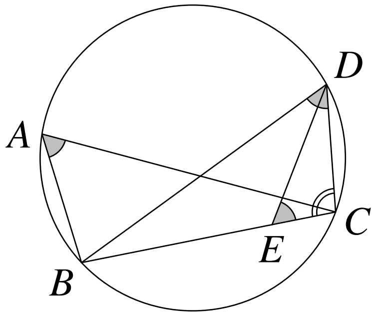
This is a very delicate question. Note that there are ‘trivial’ solutions to the equations $2b^2 = p^2 + q^2$ and $b^2 = r^2 + s^2$, namely $p = q = b$ and $r = b, s = 0$, which, of course, have nothing to do with the initial quadratics. However, there are quadratics which would give rise to these as ‘genuine’ solutions, and the conditions on the problem are necessary to ensure that this situation does not arise.

6. The diagram shows a circle with points A, B, C, D on its circumference and point E on chord BC. Given that $\angle BAC = \angle CED$ and $BC = 4 \times CE$, prove that $DB = 2 \times DE$.



Solution

Since $\angle CED = \angle CAB = \angle CDB$ (angles in the same segment) and $\angle ECD = \angle DCB$ we have $\triangle EDC \sim \triangle DBC$.



Hence $\dfrac{DB}{ED} = \dfrac{BC}{DC} = \dfrac{DC}{EC}$.

Now $\left(\dfrac{BC}{DC}\right)^2 = \dfrac{BC}{DC} \times \dfrac{DC}{EC} = \dfrac{BC}{EC} = 4$, and so $\dfrac{BC}{DC} = 2$.

Hence $\dfrac{DB}{ED} = 2$ and $DB = 2ED$ as required.